Finally, considering Eqs. (21) and (33), the nonlinear sliding robust attitude control law becomes

$$L_b = M^T(\hat{J}\wedge\Delta\dot{x} + \hat{C}\wedge\Delta x) - M^T k\,\mathrm{sgn}(s) + R_{\mathrm{BD}}L_d + \left[(R_{\mathrm{BD}}\omega_d)^\times\right]J_b\omega_{\mathrm{bd}} - J_b\left[\omega_{\mathrm{bd}}^\times\right]R_{\mathrm{BD}}\omega_d + R_{\mathrm{BD}}\left(\Delta J_d\dot{\omega}_d + \left[\omega_d^\times\right]\Delta J_d\omega_d\right) \tag{40}$$

If the selection of parameters k_i satisfy the condition (39), then control law given by Eq. (40) will satisfy the sliding condition (37) and thus lead to "perfect" tracking in the face of model uncertainty. However, it is discontinuous across the surface $s(t)$ and will result in control chattering. Chattering is undesirable because it involves extremely high control activity and may excite high-frequency dynamics that were neglected in the course of modeling. Elimination of the chattering through modification of the switching control law just derived has been discussed in Refs. 8 and 9. Based on these references, continuous approximations of the switching control law, the sgn(.) switching function, is replaced by the sat(.) nonlinear saturation function. The effect of control interpolation in the boundary layer points to assigning low-pass filter structure to the local dynamics of the variable s, thus eliminating chattering.[8,9]

Conclusions

Two attitude state controllers have been designed for tracking of large-angle maneuvers by applying a set of relative attitude kinematics and dynamics equations, where the attitude is represented by the relative modified Rodrigues parameters. The designs are the global asymptotically stable nonlinear Lyapunov controller and the robust sliding controller for attitude state tracking. Using relative attitude state equations to design tracking controller converts the tracking control problem into a regulator problem and simplifies the design procedure. The controllers implement both attitude position and angular velocity tracking. The structure of this controller can also be used for other relative attitude parameters, such as Rodrigues parameters, Euler angles, and so on. It can provide a general solution to state tracking control for rigid-body attitude.

Acknowledgment

This work was supported by NASA Goddard Space Flight Center, Greenbelt, Maryland, under Contracts NAS5-98092 and NAS5-99163.

References

[1]Wie, B., and Barba, P. M., "Quaternion Feedback for Spacecraft Large Angle Maneuvers," *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 3, 1985, pp. 360–365.

[2]Wen, J. T-Y., and Kreutz-Delgado, K. K., "The Attitude Control Problem," *IEEE Transactions on Automatic Control*, Vol. 36, No. 10, 1991, pp. 1148–1162.

[3]Crassidis, J. L., and Markley, F. L., "Sliding Mode Control Using Modified Rodrigues Parameters," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 6, 1996, pp. 1381–1383.

[4]Hall, C. D., Tsiotras, P., and Shen, H., "Tracking Rigid Body Motion Using Thrusters and Momentum Wheels," AIAA Paper 98-4471, Aug. 1998.

[5]Vadali, S. R., "Variable-Structure Control of Spacecraft Large-Angle Maneuvers," *Journal of Guidance, Control, and Dynamics*, Vol. 9, No. 2, 1986, pp. 235–239.

[6]Dwyer, T. A. W., and Ramirez, H. S., "Variable-Structure Control of Spacecraft Attitude Maneuvers," *Journal of Guidance, Control, and Dynamics*, Vol. 11, No. 3, 1988, pp. 262–269.

[7]Xing, G. Q., and Parvez, S. A., "Relative Attitude Kinematics and Dynamics Equation and Its Application to Large Angle Maneuvers and Tracking," *Proceeding of 1999 Space Control Conference*, edited by L. B. Spence, 1999, MIT Lincoln Lab., Cambridge, MA, pp. 105–114.

[8]Slotine, J. E., and Li, W., *Applied Nonlinear Control*, Prentice–Hall, Upper Saddle River, NJ, 1991, pp. 122–126.

[9]Kim, J., Kim, J., and Crassidis, J. L., "Disturbance Accommodating Sliding Mode Controller for Spacecraft Attitude Maneuvers," *Proceedings of the AAS/GSFC 13th International Symposium on Space Flight Dynamics*, Vol. 1, edited by Tom Stengle, NASA, Greenbelt, MD, 1998, pp. 119–131.

Optimal Low-Thrust Earth–Moon Targeting Strategy for *N*-Body Problem

Hui Yan* and Hongxin Wu†
Chinese Academy of Space Technology, 100080 Beijing, People's Republic of China

Introduction

THERE are many papers on determining minimum-fuel, low-thrust, Earth–moon trajectories. With regard to recent research, Kluever and Pierson[1] and Herman and Conway[2] studied optimal low-thrust, three-dimensional Earth–moon trajectories for restricted three-body problems. Their solutions, like patched conic orbits with impulsive maneuvers, are good approximations to the n-body problem. The question is how to obtain accurate solutions based on the approximations for the n-body problem. This Note presents a targeting method to adjust approximate low-thrust Earth–moon trajectories to satisfy the given requirements. We take an optimal low-thrust trajectory in the Earth's gravity field as a reference trajectory. Based on this, the optimal low-thrust Earth–moon trajectory can be obtained using a differential correction algorithm.

Optimal Low-Thrust Trajectory for Two-Body Problem

As is well known, Earth–moon trajectories are easily established by the impulsive maneuver for the n-body problem. When the trajectory is set up for the prescribed targeting parameters, the osculating ellipse at the perigee can be obtained with classical orbital elements of a_e, e_e, i_e, Ω_e, ω_e, and T_e, the semimajor axis, eccentricity, inclination, longitude of the ascending node, argument of periapsis, and time of periapsis passage, respectively. If we make the spacecraft enter the osculating elliptical orbit from the low Earth parking orbit using the optimal low-thrust vector in the single gravity field of the Earth, it is expected that we should obtain an approximate or reference optimal thrust Earth–moon orbit transfer under the control of the thrust in the n-body problem. The equations for the reference trajectory will be established in perifocal coordinates. Here the fundamental plane is coplanar with the osculating ellipse. The origin is located at Earth's center. The coordinate axes are X_ω, Y_ω, and Z_ω. The X_ω axis points toward the periapsis of the osculating ellipse; the Y_ω axis is rotated 90 deg in the direction of the elliptical motion and lies in the fundamental plane; the Z_ω axis completes the right-handed perifocal system. Let the states for the equations be $x = (r, \theta, v_r, v_\theta)^T$. They are the radial position, polar angle, radial velocity, and circumferential velocity, respectively. The equations and initial conditions are

$$\dot{r} = v_r \qquad r(0) = r_{\mathrm{LEO}} \tag{1}$$

$$\dot{\theta} = v_\theta/r \qquad \theta(0) = 0 \tag{2}$$

$$\dot{v}_r = \left(v_\theta^2/r\right) - \mu/r^2 + a_P\sin u \qquad v_r(0) = 0 \tag{3}$$

$$\dot{v}_\theta = -(v_r v_\theta/r) + a_P\cos u \qquad v_\theta(0) = \sqrt{\mu/r_{\mathrm{LEO}}} \tag{4}$$

where

$$a_P = P/(m_0 - \dot{m}t) \tag{5}$$

and the thrust angle u is the control variable, P is the thrust magnitude, a_P is the thrust acceleration, μ is the gravitational constant, m_0 is the initial spacecraft mass, $\dot{m}$ is the propellant mass flow rate, r_{LEO} is the radius of low Earth park orbit, and t is time.

Received 19 October 1999; revision received 16 September 2000; accepted for publication 14 December 2000.

*Postdoctoral Researcher, Beijing Institute of Control and Engineering; hyan@nps.navy.mil. Member AIAA.

†Professor, Beijing Institute of Control and Engineering.

The objective is to minimize the powered-flight duration

$$J = t_f \tag{6}$$

and satisfy

$$\phi_1(x_f, t_f) = a - a_e = 0 \tag{7}$$

$$\phi_2(x_f, t_f) = e - e_e = 0 \tag{8}$$

$$\phi_3(x_f, t_f) = \omega = 0 \tag{9}$$

where a, e, and ω are the semimajor axis, eccentricity, and argument of periapsis of the osculating ellipse for the low-thrust trajectory at the final time t_f.

The two-dimensional optimal low-thrust steering angle time history can be obtained using the method in Ref. 3.

Optimal Low-Thrust Earth–Moon Trajectory for *N*-Body Problem

Equations of Motion

The equations are described for the n-body problem in the geocentric equatorial coordinate frame. The fundamental plane is the Earth's equator. The X axis points toward the vernal equinox; the Y axis is 90 deg to the east in the equatorial plane; the Z axis extends through the north pole. It is assumed that only the gravitational forces and rocket thrust are considered in the equations. Neglecting the mass of the spacecraft with respect to planetary bodies, we have

$$\dot{\boldsymbol{v}}_j = -\mu_j \frac{\boldsymbol{r}_j}{\boldsymbol{r}_j^3} + \sum_{i \neq j} \mu_i \left[\frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3} - \frac{\boldsymbol{r}_i}{\boldsymbol{r}_i^3} \right] + \boldsymbol{a}_P \tag{10}$$

$$\dot{\boldsymbol{r}}_j = \boldsymbol{v}_j \tag{11}$$

where

$$\boldsymbol{a}_P = (a_x, a_y, a_z)^T \tag{12}$$

$$\boldsymbol{r}_j = (x, y, z)^T \tag{13}$$

$$\boldsymbol{v}_j = (v_x, v_y, v_z)^T \tag{14}$$

and $\boldsymbol{r}_j$ and $\boldsymbol{v}_j$ are the spacecraft position and velocity vector, respectively, $\boldsymbol{r}_i$ is the vector from the center of Earth to the ith body of the planetary system, μ_j and μ_i are the gravitational parameters of the Earth and the ith body, and $\boldsymbol{a}_P$ is the low-thrust acceleration. As the spacecraft enters the gravitational sphere of influence of the moon, similar equations will be set up in the moon-center coordinate frame with axes parallel to those of the geocentric equatorial coordinates.

Transforming the two-dimensional optimal low-thrust vector from the perifocal to the geocentric equatorial frame, we have

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \tilde{R} \begin{bmatrix} a_P \sin u \cos\theta - a_P \cos u \sin\theta \\ a_P \sin u \sin\theta + a_P \cos u \cos\theta \\ 0 \end{bmatrix} \tag{15}$$

where $\tilde{R}$ is the rotation matrix from the perifocal to the geocentric equatorial frame.[4] Using the same transformation, we obtain the initial position and velocity vectors of the spacecraft in the geocentric equatorial frame:

$$\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \tilde{R} \begin{bmatrix} r(0)\cos\theta(0) \\ r(0)\sin\theta(0) \\ 0 \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} v_x(0) \\ v_y(0) \\ v_z(0) \end{bmatrix} = \tilde{R} \begin{bmatrix} v_r(0)\cos\theta(0) - v_\theta(0)\sin\theta(0) \\ v_r(0)\sin\theta(0) + v_\theta(0)\cos\theta(0) \\ 0 \end{bmatrix} \tag{17}$$

The departure time is obtained by subtracting the powered-flight duration t_f from the epoch of the insertion point of the osculating ellipse of the impulsive maneuver Earth–moon trajectory at the perigee.

With the Eqs. (15–17) and initial time, we can integrate Eqs. (10) and (11), and the optimal low-thrust Earth–moon reference trajectories can be established for the n-body problem in the geocentric equatorial frame.

Targeting

If we assume that the osculating elliptical orbit is in the neighborhood of the actual Earth–moon trajectory by the impulsive maneuver, we can adjust the optimal low-thrust Earth–moon reference trajectory to target the goals with a differential correction algorithm. A two-dimensional differential correction process was used that allowed two independent and two dependent variables. Select a_e and e_e or ω_e as the independent variables. The two dependent variables, B-plane parameters *BT* and *BR*, are defined in the B-plane because they respond very efficiently to variations in the initial conditions.[5] The B-plane is perpendicular to the plane containing the incoming asymptote *S* of the approach hyperbola of the trajectory and the center of the moon. The vectors ***T*** and ***R*** in the B-plane and are used as axes.[5,6] The ***T*** vector is in the lunar orbit plane perpendicular to the ***S*** vector. The ***R*** vector is defined as the vertical axis in the B-plane perpendicular to vector ***T***. The *BT* and *BR* are the components along the *T* and *R* axes. The method used here sets the targeting parameters as the altitude and inclination at perilume, changes them into the B-plane parameters,[5,6] and uses Newton iteration in the differential correction process to target the goals.

Numerical Example

We select the altitude 1000 ± 1 km and inclination 90 ± 0.5 deg at perilune as the targeting parameters. The spacecraft characteristics are taken from Ref. 1. The fixed-thrust magnitude P is 2942 N, m_0 is 100,000 kg, the mass flow rate is 107.5 kg/h, and the initial low Earth parking orbit is defined by a 315-km-altitude circular orbit. When a_e and ω_e are taken as the independent variables, the iteration results are listed in Table 1, where h_m and i_m are the altitude and inclination at perilune, respectively. The first row of Table 1 is for the optimal low-thrust Earth–moon reference trajectory. The minimum time of the powered flight duration for a_e and ω_e as the independent variables is 2.2410 days. It is in good agreement with Ref. 1 for the Earth escape stage (2.24 days) for the restricted three-body problem dynamics. The optimal, low-thrust Earth–moon trajectory in the geocentric equatorial coordinate frame is shown in Fig. 1; the total flight time is about 7.075 days.

Table 1 Iteration results for a_e and ω_e as the independent variables

No.	a_e, km	ω_e, rad	*BT*, km	*BR*, km	h_m, km	i_m, deg
1	218969.4	−2.684964	−83.86	235.84	853.82	90.84
2	218935.1	−2.685973	−27.35	−38.56	1024.60	90.44
3	218923.0	−2.685942	−13.08	−13.18	1008.60	90.34
4	218917.2	−2.685943	−3.04	−4.25	1002.98	90.28
5	218915.8	−2.685940	−1.42	−1.46	1001.23	90.27
6	218915.2	−2.685941	−0.68	−0.70	1000.75	90.26

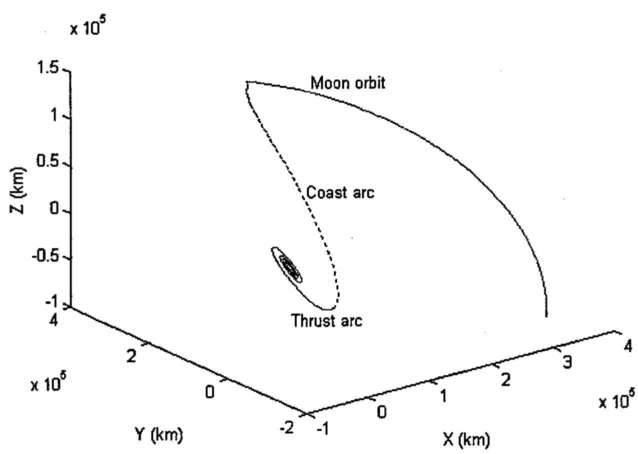

Fig. 1 Optimal low-thrust Earth–moon trajectory.

Conclusions

The paper has studied the optimal low-thrust Earth–moon targeting strategy for the n-body problem. First, the optimal, low-thrust reference Earth–moon trajectory is established. Then, taking the osculating orbital elements as the independent variables and adjusting the optimal low-thrust, reference Earth–moon trajectory with the differential correction algorithm, we achieve the targeting parameters. The numerical results confirm the algorithm works successfully.

Acknowledgment

This work was funded by the Chinese National Science Foundation.

References

[1]Kluever, C. A., and Pierson, B. L., "Optimal Low-Thrust Three-Dimensional Earth–Moon Trajectories," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 4, 1995, pp. 830–837.

[2]Herman, A. L., and Conway, B. A., "Optimal, Low-Thrust, Earth–Moon Orbit Transfer," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 1, 1998, pp. 141–147.

[3]Yan, H., and Wu, H., "Initial Adjoint-Variable Guess Technique and Its Application in Optimal Orbital Transfer," *Journal of Guidance, Control, and Dynamics*, Vol. 22, No. 3, 1999, pp. 490–492.

[4]Bate, R. R., Mueller, D. D., and White, J. E., *Fundamentals of Astrodynamics*, Dover, New York, 1971, pp. 80–83.

[5]Carrico, J., Hooper, H. L., Roszman, L., and Gramling, C., "Rapid Design of Gravity Assist Trajectories," *Proceedings of the ESA Symposium on Spacecraft Flight Dynamics*, European Space Agency, Darmstadt, Germany, 1991, pp. 427–434.

[6]Folta, D. C., and Sharer, P. J., "Multiple Lunar Flyby Targeting for the WIND Mission," *Spaceflight Mechanics 1996*, Vol. 93, Advances in the Astronautical Sciences, Univelt, San Diego, CA, 1996, pp. 53–67.

Range, Range Rate, and Acceleration Computation for Inclined Geosynchronous Orbit

Kar W. Yung* and Peter B. Landecker*
Hughes Space and Communications Company
Los Angeles, California 90009

Nomenclature

i = inclination angle of the geosynchronous satellite
$\boldsymbol{R}$ = position vector of the geosynchronous satellite with respect to a ground location
$\boldsymbol{R}_G$ = position vector of the ground location relative to the center of the Earth
$\boldsymbol{R}_S$ = position vector of the geosynchronous satellite relative to the center of the Earth
$\dot{\boldsymbol{R}}$ = velocity vector of the geosynchronous satellite with respect to a ground location
$\dot{\boldsymbol{R}}_S$ = velocity vector of the geosynchronous satellite
r = range of the geosynchronous satellite with respect to a ground location
r_G = radius of Earth, approximately 6378 km
r_S = geosynchronous orbital radius, approximately 42,165 km
$\dot{r}$ = range rate of the geosynchronous satellite with respect to a ground location
$\dot{r}_{\max}$ = maximum range rate for all possible ground locations within the field of view of the satellite
$\dot{r}_{\max 1}$ = maximum range rate for one single ground location
t = time elapsed since passage of satellite from the ascending node
η = latitude
θ = central angle between satellite and ground unit position vectors
λ = longitudinal displacement from the ascending node
ω = angular rate of Earth as well as geosynchronous orbit

Received 3 December 1999; revision received 31 July 2000; accepted for publication 1 November 2000.

*Senior Scientist, System Products, P.O. Box 92919.

Introduction

THIS Note derives closed-form solutions for range, range rate, and range acceleration between an inclined geosynchronous satellite and locations on the ground surface. The motivation of this exercise is to avoid the complexities of numerical analyses (for example, in FORTRAN) by instead using a simple algebraic formula solution amenable for easy use (for example, in spreadsheets). A practical reason for why this calculation is useful is the need to know the maximum Doppler shift for devices for communication gateways as part of the filter bandpass design specification.

The coordinate system is defined. Each ground location is specified by its latitude and longitudinal displacement from the ascending node. Closed-form equations for range, range rate, and range acceleration are then given. An expression for maximum possible range rate is also derived. The resulting solutions are simple algebriac expressions.

These results are potentially useful for several commercial satellite communication programs that employ constellations of inclined geosynchronous Earth orbit (GEO) satellites. This calculation has not been previously published[1–3] (Wertz, J. R., private communication).

In our model, we assume the Earth is a perfect sphere and it rotates at the same angular rate as the Earth. The vectors $\boldsymbol{R}$, $\dot{\boldsymbol{R}}$, $\boldsymbol{R}_G$, $\boldsymbol{R}_S$, and $\dot{\boldsymbol{R}}_S$ are defined in a coordinate system that rotates with the Earth and are shown in Fig. 1. Its x axis points toward the ascending node and its z axis points toward the north pole.

Range and Its Derivatives

The position vectors of the ground location and the satellite are as follows:

$$\boldsymbol{R}_G = r_G \begin{pmatrix} \cos\lambda\cos\eta \\ \sin\lambda\cos\eta \\ \sin\eta \end{pmatrix} \tag{1}$$

$$\boldsymbol{R}_S = r_S \begin{pmatrix} \frac{1}{2}\{1+\cos(i)+[1-\cos(i)]\cos(2\omega t)\} \\ -\frac{1}{2}[1-\cos(i)]\sin(2\omega t) \\ \sin(i)\sin(\omega t) \end{pmatrix} \tag{2}$$

$$\boldsymbol{R} = \boldsymbol{R}_S - \boldsymbol{R}_G \tag{3}$$

The range between the ground location and the satellite is given by

$$r = \sqrt{r_S^2 + r_G^2 - 2r_S r_G A} = r_S\sqrt{1 + (r_G/r_S)^2 - 2(r_G/r_S)A} \tag{4}$$

where

$$A = \tfrac{1}{2}\cos\lambda\cos\eta[1+\cos(i)] + \sin\eta\sin(i)\sin(\omega t) + \tfrac{1}{2}\cos\eta[1-\cos(i)]\cos(\lambda+2\omega t) \tag{5}$$

The ratio of ranges is defined as follows:

$$\beta = r_G/r_S \tag{6}$$

Because the satellite is always above the surface of the Earth,

$$0 < \beta < 1 \tag{7}$$

The range is, therefore, given by

$$r = r_S\sqrt{1+\beta^2-2\beta A} \tag{8}$$

The velocity vector of the satellite is given by

$$\dot{\boldsymbol{R}} = \dot{\boldsymbol{R}}_S = r_S\omega \begin{pmatrix} -\frac{1}{2}[1-\cos(i)]\sin(2\omega t) \\ \frac{1}{2}\{1+\cos(i)-[1-\cos(i)]\cos(2\omega t)\} \\ \sin(i)\cos(\omega t) \end{pmatrix} \tag{9}$$